Enumerating minimal dominating sets and variants in chordal bipartite graphs

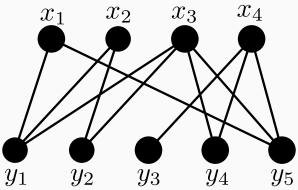
Emanuel Castelo
Oscar Defrain
Guilherme C. M. Gomes

Aix-Marseille Université, France
Universidade Federal de Minas Gerais, Brazil

Chordal bipartite: definition

Definition (Chordal bipartite)

A bipartite graph is **chordal bipartite** if it does not contain induced cycles of length at least six.

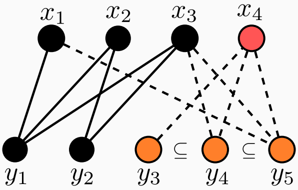


Chordal bipartite: elimination ordering

Definition (Weak-simplicial)

A vertex $v \in V(G)$ is **weak-simplicial** if

- $N(v)$ is an independent set;
- for every pair $x, y \in N(v)$, either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$ [1].



[1] x and y are said to be **comparable**.

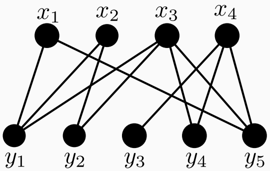
Chordal bipartite: elimination ordering

Definition (Weak-simplicial elimination ordering)

An ordering $v_1, \ldots, v_n$ of $V(G)$ is said to be a **weak-simplicial elimination ordering** if, for all $i \in [n]$, v_i is weak-simplicial in $G_i = G[\{v_i, \ldots, v_n\}]$.

Theorem ([1])

A graph is chordal bipartite if and only if it admits a weak-simplicial elimination ordering.



$(x_2, x_1, y_1, y_2, x_4, y_3, y_4, x_3, y_5)$ is a weak-simplicial ordering of G.

Domination problems

Definition (Dominating Set)

A set $D \subseteq V(G)$ is a dominating set of G if $N[D] = V(G)$.

Definition (Total Dominating Set)

A set $D \subseteq V(G)$ is a total dominating set of G if $N(D) = V(G)$.

Definition (Connected Dominating Set)

A set $D \subseteq V(G)$ is a connected dominating set of G if $N[D] = V(G)$ and $G[D]$ is connected.

In all cases, D is said to be minimal if it is inclusion-wise minimal.

Enumeration variants

Minimal Dominating Set Enumeration (MinDom·Enum)

Input: A graph G.

Output: All minimal dominating sets of G.

Minimal Connected Dominating Set Enumeration (CMinDom·Enum)

Input: A graph G.

Output: All minimal connected dominating sets of G.

Minimal Total Dominating Set Enumeration (TMinDom·Enum)

Input: A graph G.

Output: All minimal total dominating sets of G.

State-of-the-art and directions

What do we know?

- MinDom·Enum is in **IncP** on chordal bipartite graphs [2];
- TMinDom·Enum is in **DelayP** on chordal bipartite graphs [2]; and
- CMinDom·Enum is hard [3].

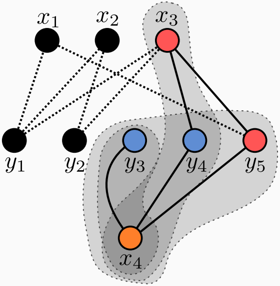
- **Question 1:** Is MinDom·Enum in **DelayP** on chordal bipartite graphs?
- **Question 2:** Is CMinDom·Enum in **TotalP** on chordal bipartite graphs?

The algorithm: general idea

Observation

Minimal dominating sets in a graph corresponds to **minimal transversals** of the closed neighborhood hypergraph.

- Fix an ordering $(v_1, \ldots, v_n)$ of $V(G)$;
- $\mathcal{H}_i :=$ hypergraph of closed neighborhoods of G included in $\{v_1, \ldots, v_i\}$;
- Extend a $T \in Tr(\mathcal{H}_i)$ to a $T' \in Tr(\mathcal{H}_{i+1})$.

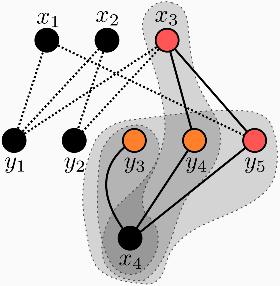


The algorithm: general idea

Observation

Minimal dominating sets in a graph corresponds to **minimal transversals** of the closed neighborhood hypergraph.

- Fix an ordering $(v_1, \ldots, v_n)$ of $V(G)$;
- $\mathcal{H}_i :=$ hypergraph of closed neighborhoods of G included in $\{v_1, \ldots, v_i\}$;
- Extend a $T \in Tr(\mathcal{H}_i)$ to a $T' \in Tr(\mathcal{H}_{i+1})$.

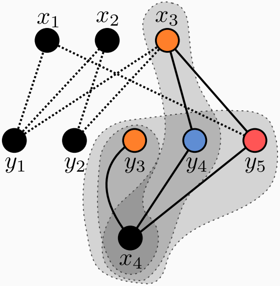


The algorithm: general idea

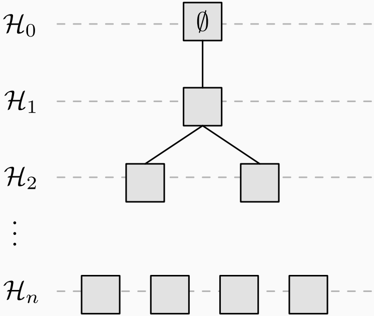
Observation

Minimal dominating sets in a graph corresponds to **minimal transversals** of the closed neighborhood hypergraph.

- Fix an ordering $(v_1, \ldots, v_n)$ of $V(G)$;
- $\mathcal{H}_i :=$ hypergraph of closed neighborhoods of G included in $\{v_1, \ldots, v_i\}$;
- Extend a $T \in Tr(\mathcal{H}_i)$ to a $T' \in Tr(\mathcal{H}_{i+1})$.



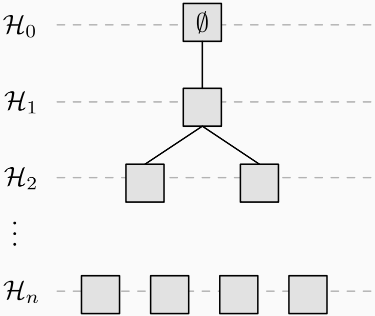
How to extend 101: the sequential method



What properties do we want? A search tree!

- No cycles; and
- No leaves of height $i < n$.

How to extend 101: the sequential method



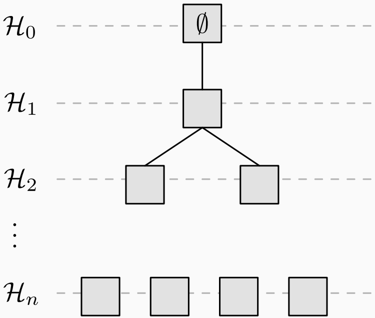
What properties do we want? A search tree!

- No cycles; and $\star$
- No leaves of height $i < n$.

Parent of $T \in Tr(\mathcal{H}_{i+1})$ ($\star$)

Repetedly remove the smallest vertex $v \in T$ with no private neighbors in $\mathcal{H}_i$.

How to extend 101: the sequential method



What properties do we want? A search tree!

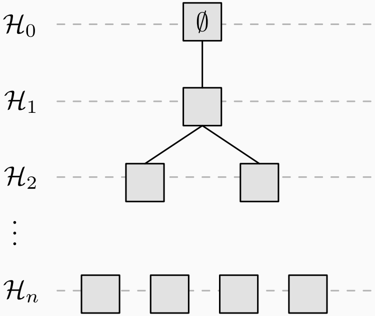
- No cycles; and
- No leaves of height $i < n$. $\star$

Ensuring children for $i \in [n-1]$ ($\star$)

If $T^\star \in Tr(\mathcal{H}_i)$, $i \in [n-1]$, then either:

- $T^\star \in Tr(\mathcal{H}_{i+1})$ and is its own parent; or
- $T^\star \cup \{v_{i+1}\} \in Tr(\mathcal{H}_{i+1})$ and $T^\star$ is its parent.

How to extend 101: the sequential method



- Δ_{i+1} := subset of $\mathcal{H}_{i+1}$ not hit by $T \in Tr(\mathcal{H}_i)$;

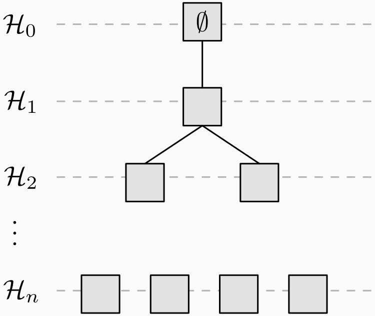
Lemma

If T is a child of $T^\star$, then $T = T^\star \cup X$, where $X \in Tr(\Delta_{i+1})$.

Observation

If $|Tr(\Delta_{i+1})|$ is polynomial, then we have **DelayP** + **PSPACE**!

How to extend 101: the sequential method



- Δ_{i+1} := subset of $\mathcal{H}_{i+1}$ not hit by $T \in Tr(\mathcal{H}_i)$;
- **Bad news:** $|Tr(\Delta_{i+1})|$ can be exponential in general;
- **Good news:** polynomial for chordal bipartite graphs!

Lemma

If T is a child of $T^\star$, then $T = T^\star \cup X$, where $X \in Tr(\Delta_{i+1})$.

Observation

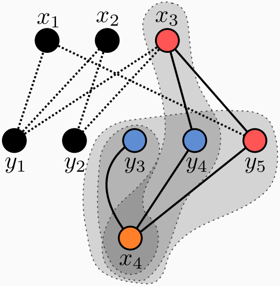
If $|Tr(\Delta_{i+1})|$ is polynomial, then we have **DelayP** + **PSPACE**!

MinDom·Enum

Observation

If v is weak-simplicial, then $G[N[v] \cup N^2(v)]$ is **bipartite chain**.

- $B \coloneqq \{u \in N(v_{i+1}) \mid N[u] \in \Delta_{i+1}\}$;
- $R \coloneqq \left(\bigcup_{H \in \Delta_{i+1}} H\right) \setminus B$.

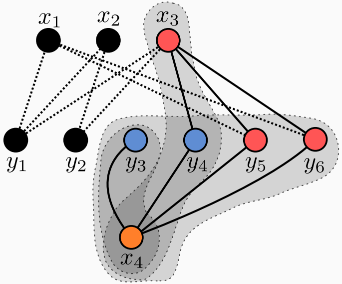


MinDom·Enum

Lemma

Let $T \in Tr(\Delta_{i+1})$*. Then*

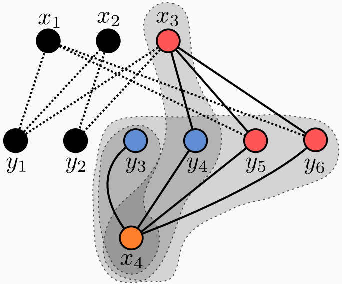
- $|T \cap R \cap N(v_{i+1})| \leq 1$*; and*
- $|T \cap N^2(v_{i+1})| \leq 1$*.*



Lemma

Let $T \in Tr(\Delta_{i+1})$. Then exactly one of the following holds:

- $T = \{v_{i+1}\}$;
- $T \subseteq B$, *in which case* $T = B$;
- $T \subseteq R$, *in which case* $|T| \leq 2$;
- $T = \{r\} \cup (B \setminus N(r))$ *for some* $r \in N^2(v_{i+1})$.



Results

Theorem ([4])

MinDom·Enum *is in* **DelayP** + **PSPACE** *on chordal bipartite graphs.*

Theorem ([4])

CMinDom·Enum *is in* **IncP** *on chordal bipartite graphs.*

Theorem ([4])

The sequential method is **NP***-complete for minimal dominating sets even if restricted bipartite graphs.*

Theorem ([4, 5])

CMinDom·Enum *is* MinTrans·Enum*-hard even if restricted to bipartite graphs.*

References

[1] Kazuhiro Kurita et al. **"An Efficient Algorithm for Enumerating Chordal Bipartite Induced Subgraphs in Sparse Graphs".** In: *International Workshop on Combinatorial Algorithms*. Springer. 2019, pp. 339–351.

[2] Petr A Golovach et al. **"Enumerating minimal dominating sets in chordal bipartite graphs".** In: *Discrete Applied Mathematics* 199 (2016), pp. 30–36.

[3] Mamadou Moustapha Kanté et al. **"On the enumeration of minimal dominating sets and related notions".** In: *SIAM Journal on Discrete Mathematics* 28.4 (2014), pp. 1916–1929.

[4] Emanuel Castelo, Oscar Defrain, and Guilherme Gomes. **"Enumerating minimal dominating sets and variants in chordal bipartite graphs".** In: *arXiv preprint arXiv:2502.14611* (2025).

[5] Yasuaki Kobayashi et al. **“Enumerating minimal vertex covers and dominating sets with capacity and/or connectivity constraints”.** In: *Algorithms* 18.2 (2025), p. 112.