Generating minimal redundant and maximal irredundant sets in graphs

Emanuel Castelo

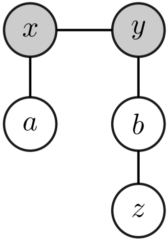
June 20, 2026

Basic definitions

Let G be a graph.

A set $S \subseteq V(G)$ is said to be irredundant if for each $x \in X$ it follows that $N[x] \setminus N[X \setminus x]$ is non-empty.

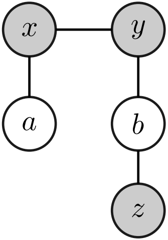
A set $S \subseteq V(G)$ is said to be redundant if it is not irredundant.



Let G be a graph.

A set $S \subseteq V(G)$ is said to be irredundant if for each $x \in X$ it follows that $N[x] \setminus N[X \setminus x]$ is non-empty.

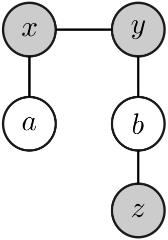
A set $S \subseteq V(G)$ is said to be redundant if it is not irredundant.



Let $\mathcal{H}$ be a hypergraph.

A set $S \subseteq V(\mathcal{H})$ is said to be irredundant if for each $x \in X$ it follows that $\{E \in \mathcal{E}(\mathcal{H}) \mid x \in E \wedge E \cap (X \setminus x) = \emptyset\}$ is non-empty.

A set $S \subseteq V(\mathcal{H})$ is said to be redundant if it is not irredundant.



Let MaxIrr(G) and MinRed(G) denote the family of maximal irredundant and minimal redundant sets of G, respectively.

GraphMirr·Enum
In: A graph G.
Out: MaxIrr(G).

GraphMred·Enum
In: A graph G.
Out: MinRed(G).

Let MaxIrr(G) and MinRed(G) denote the family of maximal irredundant and minimal redundant sets of G, respectively.

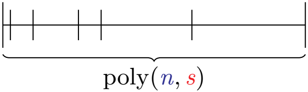
GraphMirr·Enum
In: A graph G.
Out: MaxIrr(G).

GraphMred·Enum
In: A graph G.
Out: MinRed(G).

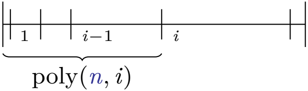
HypMirr·Enum and HypMred·Enum when looking at hypergraphs!

Let n be the size of the instance and s the number of solutions.

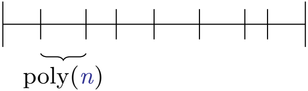
Output polynomial (**OutputP**)



Incremental polynomial (**IncP**)



Polynomial delay (**DelayP**)



[Uno '15]

Is there an output-polynomial-time algorithm for HypMirr·Enum?

[Uno '15]

Is there an output-polynomial-time algorithm for HypMirr·Enum?

Theorem (Boros, Makino, '16)

There is no output-polynomial-time algorithm for HypMirr·Enum *and* HypMred·Enum *unless* $\mathbf{P} = \mathbf{NP}$*, even when restricted to hypergraphs of dimension at most three.*

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMirr·Enum:

- Bounded degree graphs [Boros, Makino, '24];

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMirr·Enum:

- Bounded degree graphs [Boros, Makino, '24];
- d-degenerate graphs [Conte, Kanté, Marino, Uno, '19];

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMirr·Enum:

- Bounded degree graphs [Boros, Makino, '24];
- d-degenerate graphs [Conte, Kanté, Marino, Uno, '19];
- Permutation graphs [Blind, Creignou, Olive, '21];

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMirr·Enum:

- Bounded degree graphs [Boros, Makino, '24];
- d-degenerate graphs [Conte, Kanté, Marino, Uno, '19];
- Permutation graphs [Blind, Creignou, Olive, '21];
- Circular-permutation graphs [Blind, Creignou, Olive, '21];

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMirr·Enum:

- Bounded degree graphs [Boros, Makino, '24];
- d-degenerate graphs [Conte, Kanté, Marino, Uno, '19];
- Permutation graphs [Blind, Creignou, Olive, '21];
- Circular-permutation graphs [Blind, Creignou, Olive, '21];
- Strongly orderable graphs [Castelo, Chalopin, Defrain, Vilmin, '26].

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMred·Enum:

- bounded degree graphs [Boros, Makino, '24]; .

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMred·Enum:

- bounded degree graphs [Boros, Makino, '24];
- (C_3, C_5, C_6, C_8)-free graphs [Castelo, Chalopin, Defrain, Vilmin, '26]; .

Is there an output-polynomial-time algorithm for GraphMirr·Enum or GraphMred·Enum?

Known results for GraphMred·Enum:

- bounded degree graphs [Boros, Makino, '24];
- (C_3, C_5, C_6, C_8)-free graphs [Castelo, Chalopin, Defrain, Vilmin, '26];
- (C_3, C_5, C_6)-free graphs* [Castelo, Chalopin, Defrain, Vilmin, '26].

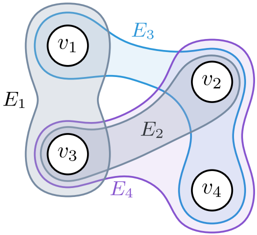
A hardness proof

Theorem (Castelo, Chalopin, Defrain, Vilmin, '26)

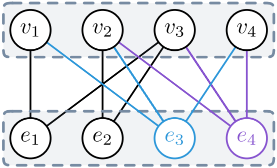
There is no output-polynomial-time algorithm for GraphMred·Enum *unless* $\mathbf{P} = \mathbf{NP}$, *even when restricted to split and co-bipartite graphs.*

Theorem (Castelo, Chalopin, Defrain, Vilmin, '26)

There is no output-polynomial-time algorithm for GraphMred·Enum *unless* $\mathbf{P} = \mathbf{NP}$*, even when restricted to split and co-bipartite graphs.*



$\mathcal{H}$

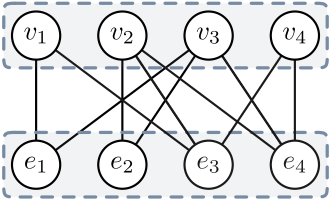


$C(\mathcal{H})$

Claim

MinRed$(C(\mathcal{H})) = \mathcal{X} \cup \mathcal{R}$, where

- $\mathcal{X} = \{R \mid R \in \text{MinRed}(C(\mathcal{H})),\ |R| \leq 4\}$; and
- $\mathcal{R} = \{R \mid R \in \text{MinRed}(\mathcal{H}),\ |R| \geq 5\}$.

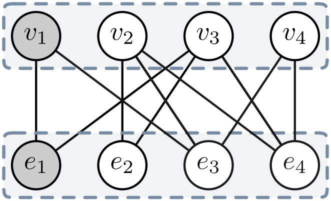


$C(\mathcal{H})$

Claim

MinRed$(C(\mathcal{H})) = \mathcal{X} \cup \mathcal{R}$, where

- $\mathcal{X} = \{R \mid R \in \text{MinRed}(C(\mathcal{H})),\ |R| \leq 4\}$; and
- $\mathcal{R} = \{R \mid R \in \text{MinRed}(\mathcal{H}),\ |R| \geq 5\}$.



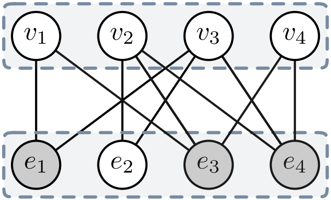
$C(\mathcal{H})$

- If $R \cap V \neq \emptyset$ and $R \cap U \neq \emptyset$, then $|R| \leq 3$

Claim

MinRed$(C(\mathcal{H})) = \mathcal{X} \cup \mathcal{R}$, where

- $\mathcal{X} = \{R \mid R \in \text{MinRed}(C(\mathcal{H})),\ |R| \leq 4\}$; and
- $\mathcal{R} = \{R \mid R \in \text{MinRed}(\mathcal{H}),\ |R| \geq 5\}$.



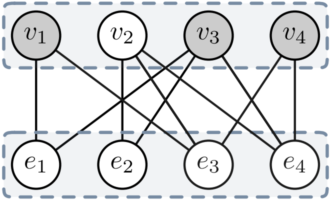
$C(\mathcal{H})$

- If $R \cap V \neq \emptyset$ and $R \cap U \neq \emptyset$, then $|R| \leq 3$
- If $R \subseteq U$, then $|R| \leq 4$

Claim

$\mathsf{MinRed}(C(\mathcal{H})) = \mathcal{X} \cup \mathcal{R}$, where

- $\mathcal{X} = \{R \mid R \in \mathsf{MinRed}(C(\mathcal{H})),\ |R| \leq 4\}$; and
- $\mathcal{R} = \{R \mid R \in \mathsf{MinRed}(\mathcal{H}),\ |R| \geq 5\}$.

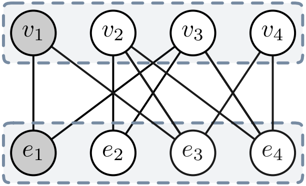


$C(\mathcal{H})$

- If $R \cap V \neq \emptyset$ and $R \cap U \neq \emptyset$, then $|R| \leq 3$
- If $R \subseteq U$, then $|R| \leq 4$
- If $R \subseteq V$, then $R \in \mathsf{MinRed}(\mathcal{H})$

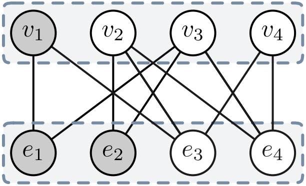
Excluding small cycles

Given a graph G, a vertex $x \in V(G)$, and a set $R \in \text{MinRed}(G)$.



- $\text{priv}(x, R) := \{y \in N[x] \setminus N[R \setminus x]\}$
  e.g. $\text{priv}(e_1, \{v_1, e_1\}) = \{e_2, e_4\}$
- $\text{red}(R) := \{x \in R \mid \text{priv}(x, R) = \emptyset\}$
  e.g. $\text{red}(\{v_1, e_1\}) = \emptyset$

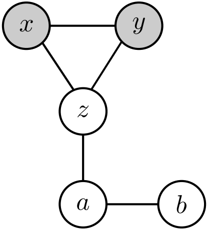
Given a graph G, a vertex $x \in V(G)$, and a set $R \in \text{MinRed}(G)$.



- $\text{priv}(x, R) := \{y \in N[x] \setminus N[R \setminus x]\}$
  e.g. $\text{priv}(e_1, \{v_1, e_1, e_2\}) = \emptyset$
- $\text{red}(R) := \{x \in R \mid \text{priv}(x, R) = \emptyset\}$
  e.g. $\text{red}(\{v_1, e_1, e_2\}) = \{e_1, e_2\}$

Lemma

For any two distinct vertices $x, y \in R$ *such that* $x \in \mathsf{red}(R)$ *there exists a vertex* $z \in \mathsf{priv}(x, R \setminus \{y\}) \cap \mathsf{priv}(y, R \setminus \{x\})$.



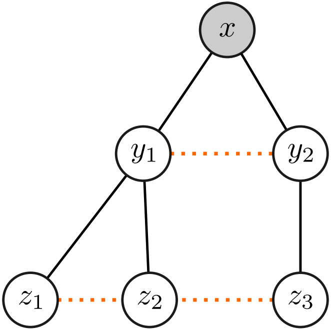
Lemma

$R \subseteq N^2[x]$ *for every* $x \in \mathsf{red}(R)$.

Let G be a (C_3, C_5)-free graph.

Lemma

For every vertex $x \in V(G)$, the graph induced by $N^2[x]$ is bipartite.

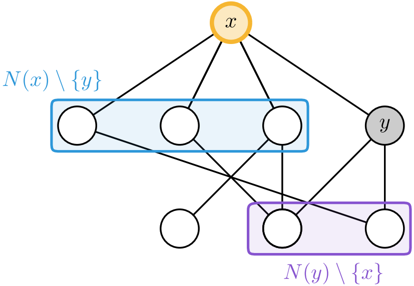


Lemma

Let $x \in \mathrm{red}(R)$*. If* $y \in R$ *is adjacent to* x *and* $N(y) \neq \{x\}$*, then at most one of the following holds:*

- $R \cap (N(y) \setminus \{x\}) \neq \emptyset$*; or*
- $R \cap (N(x) \setminus \{y\}) \neq \emptyset$*.*

Moreover, if $y \in \mathrm{red}(R)$ *then exactly one of the items is satisfied.*

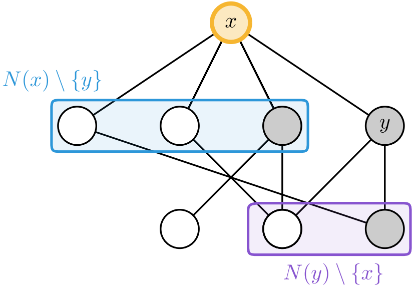


Lemma

Let $x \in \text{red}(R)$. *If* $y \in R$ *is adjacent to* x *and* $N(y) \neq \{x\}$, *then at most one of the following holds:*

- $R \cap (N(y) \setminus \{x\}) \neq \emptyset$; *or*
- $R \cap (N(x) \setminus \{y\}) \neq \emptyset$.

Moreover, if $y \in \text{red}(R)$ *then exactly one of the items is satisfied.*

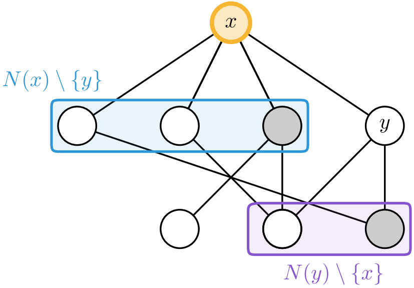


Lemma

Let $x \in \mathrm{red}(R)$. If $y \in R$ is adjacent to x and $N(y) \neq \{x\}$, then at most one of the following holds:

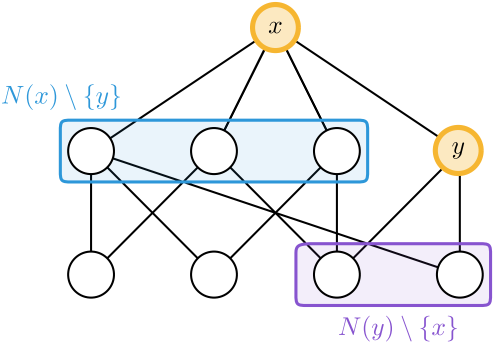
- $R \cap (N(y) \setminus \{x\}) \neq \emptyset$; *or*
- $R \cap (N(x) \setminus \{y\}) \neq \emptyset$.

Moreover, if $y \in \mathrm{red}(R)$ then exactly one of the items is satisfied.



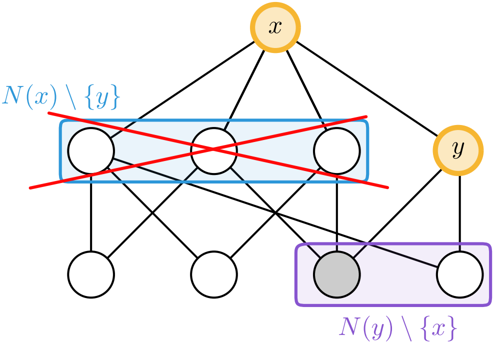
Lemma

Suppose there exists an edge xy such that $x, y \in \text{red}(R)$. Then either $R = N[y]$ or $R = N[x]$.



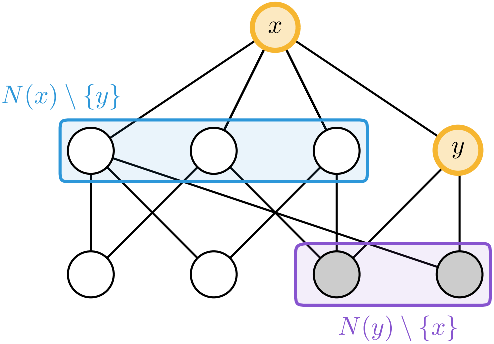
Lemma

Suppose there exists an edge xy such that $x, y \in \text{red}(R)$. Then either $R = N[y]$ or $R = N[x]$.



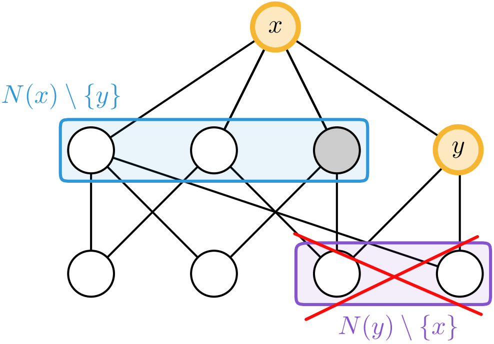
Lemma

Suppose there exists an edge xy such that $x, y \in \text{red}(R)$. Then either $R = N[y]$ or $R = N[x]$.



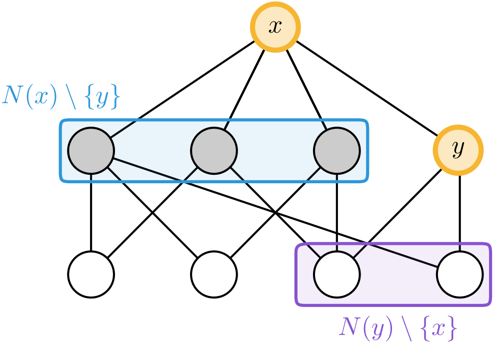
Lemma

Suppose there exists an edge xy *such that* $x, y \in \text{red}(R)$*. Then either* $R = N[y]$ *or* $R = N[x]$*.*



Lemma

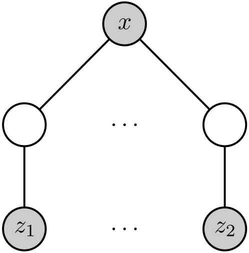
Suppose there exists an edge xy such that $x, y \in \mathsf{red}(R)$. Then either $R = N[y]$ or $R = N[x]$.



Time to forbid C_6!

Lemma

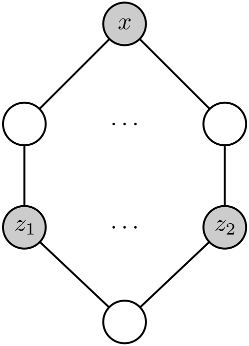
Let $x \in \mathsf{red}(R)$ *and* z_1, z_2 *be two distinct vertices in* $R \cap N^2(x)$. *Then* $(N(z_1) \cap N(z_2)) \setminus N(x) = \emptyset$.



Time to forbid C_6!

Lemma

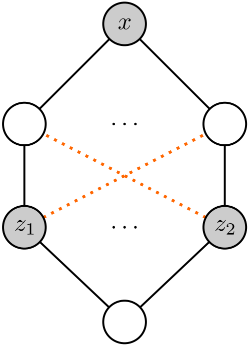
Let $x \in \mathrm{red}(R)$ *and* z_1, z_2 *be two distinct vertices in* $R \cap N^2(x)$. *Then* $(N(z_1) \cap N(z_2)) \setminus N(x) = \emptyset$.



Time to forbid C_6!

Lemma

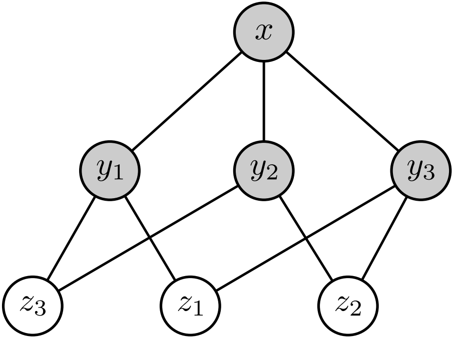
Let $x \in \mathrm{red}(R)$ and z_1, z_2 be two distinct vertices in $R \cap N^2(x)$. Then $(N(z_1) \cap N(z_2)) \setminus N(x) = \emptyset$.



Lemma

If $R \in \text{MinRed}(G)$ *and* $x \in \text{red}(R)$*, then:*

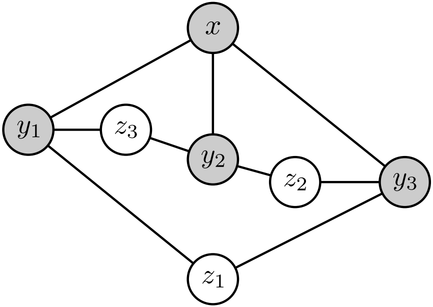
- $|\text{red}(R) \cap N(x)| \leq 2$*; and*
- $|\text{red}(R) \cap N^2(x)| \leq 1$*.*



Lemma

If $R \in \mathsf{MinRed}(G)$ *and* $x \in \mathsf{red}(R)$, *then:*

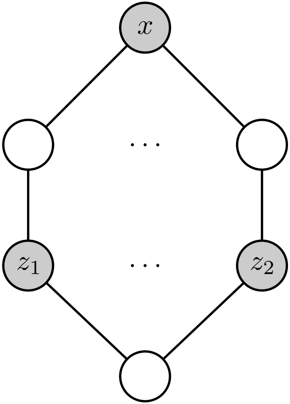
- $|\,\mathsf{red}(R) \cap N(x)| \leq 2$; *and*
- $|\,\mathsf{red}(R) \cap N^2(x)| \leq 1$.



Lemma

If $R \in \mathrm{MinRed}(G)$ *and* $x \in \mathrm{red}(R)$*, then:*

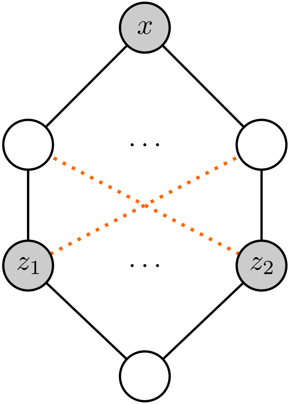
- $|\,\mathrm{red}(R) \cap N(x)| \leq 2$*; and*
- $|\,\mathrm{red}(R) \cap N^2(x)| \leq 1$*.*



Lemma

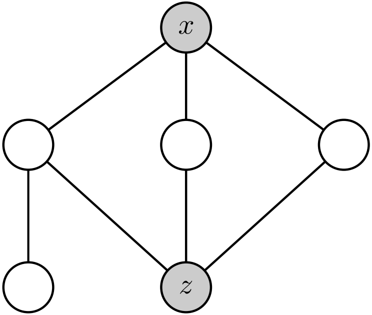
If $R \in \mathsf{MinRed}(G)$ *and* $x \in \mathsf{red}(R)$*, then:*

- $|\mathsf{red}(R) \cap N(x)| \leq 2$*; and*
- $|\mathsf{red}(R) \cap N^2(x)| \leq 1$.



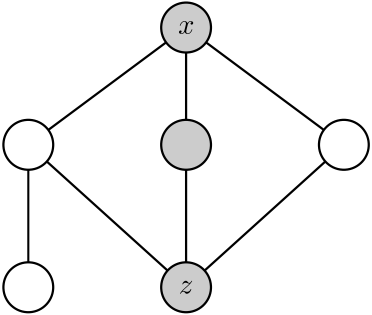
Lemma

If $R \in \mathsf{MinRed}(G)$ *such that* $x \in \mathsf{red}(R)$ *and* $|\,\mathsf{red}(R) \cap N^2(x)| = 1$*, then* $|R \cap N(x)| = 1$*, and so* $|R| = 3$*.*



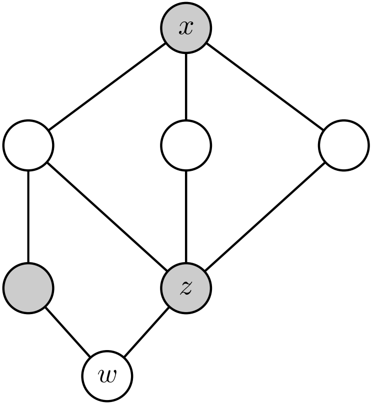
Lemma

If $R \in \mathsf{MinRed}(G)$ *such that* $x \in \mathsf{red}(R)$ *and* $|\,\mathsf{red}(R) \cap N^2(x)| = 1$*, then* $|R \cap N(x)| = 1$*, and so* $|R| = 3$*.*



Lemma

If $R \in \text{MinRed}(G)$ *such that* $x \in \text{red}(R)$ *and* $|\,\text{red}(R) \cap N^2(x)| = 1$, *then* $|R \cap N(x)| = 1$, *and so* $|R| = 3$.

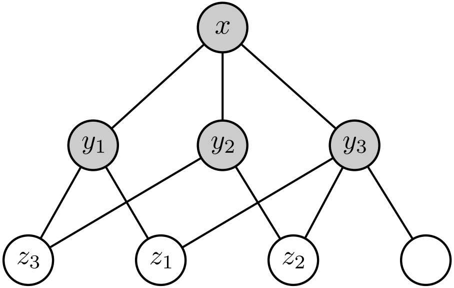


Corollary

If G *is* (C_3, C_5, C_6)*-free and* $R \in \text{MinRed}(G)$, *then* R *contains at most 3 redundant vertices.*

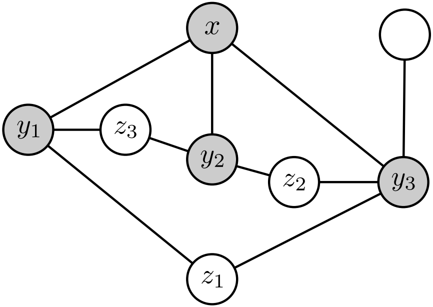
Lemma

If $R \in \mathsf{MinRed}(G)$ *such that* $x \in \mathsf{red}(R)$ *and* $|\,\mathsf{red}(R) \cap N(x)| = 2$*, then* $|R| = 3$*.*



Lemma

If $R \in \mathsf{MinRed}(G)$ *such that* $x \in \mathsf{red}(R)$ *and* $|\,\mathsf{red}(R) \cap N(x)| = 2$*, then* $|R| = 3$*.*



Corollary

If $R \in \mathsf{MinRed}(G)$ *contains 3 redundant vertices, then* $|R| = 3$*.*

Corollary

If $R \in \mathsf{MinRed}(G)$ *contains 2 redundant vertices, then either:*

- $R = N[x]$ *for some* $x \in \mathsf{red}(R)$*; or*
- $|R| = 3$*.*

The case of a single redundant vertex

Lemma

Let $x \in V(G)$ and $y \in N(x)$. The following are equivalent:

(a) $R \in \mathsf{MinRed}(G)$ *where* $\mathsf{red}(R) = \{x\}$ *and* $N(x) \cap R = \{y\}$.

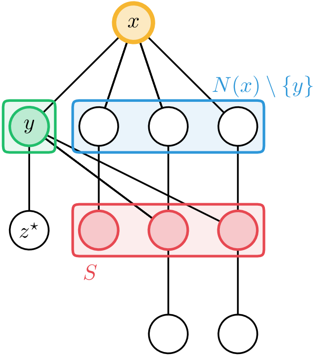
(b) $R = S \cup \{x, y\}$ *where:*

   i) $S \subseteq N^2(x)$;

   ii) *The vertex y has a private neighbor with respect to $S \cup \{x\}$;*

   iii) *For every $z \in S$ satisfying $N(z) \subseteq N(x)$ it follows that $z \notin N(y)$; and*

   iv) *S minimally dominates $N(x) \setminus \{y\}$;*



Lemma

Let $x \in V(G)$ and $Y \subseteq N(x)$ such that $|Y| \geq 2$. The following are equivalent:

(a) $R \in \mathrm{MinRed}(G)$ *where* $\mathrm{red}(R) = \{x\}$ *and* $N(x) \cap R = Y$.

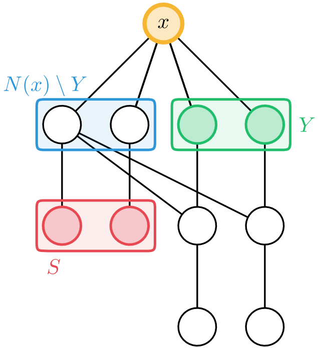
(b) $R = S \cup Y \cup \{x\}$ *such that:*

i) $S \subseteq N^2(x)$;

ii) $S \cap N(Y) = \emptyset$;

iii) *Each vertex* $y \in Y$ *has a private neighbor with respect to* $Y \cup \{x\}$;

iv) S *minimally dominates* $N(x) \setminus Y$.



An algorithm

[Kanté, Khoshkhah, Pourmoradnasseri, '18]
Minimal red-blue dominating set is tractable on (C_6, C_8)-free bipartite graphs.

Overview of the algorithm:

[Kanté, Khoshkhah, Pourmoradnasseri, '18]
Minimal red-blue dominating set is tractable on (C_6, C_8)-free bipartite graphs.

Overview of the algorithm:

- If $|\text{red}(R)| \geq 2$, then we can list by brute-force in polynomial time!

[Kanté, Khoshkhah, Pourmoradnasseri, '18]
Minimal red-blue dominating set is tractable on (C_6, C_8)-free bipartite graphs.

Overview of the algorithm:

- If $|\text{red}(R)| \geq 2$, then we can list by brute-force in polynomial time!
- If $|\text{red}(R)| = |R \cap N(x)| = 1$, then we try all triplets (x, y, z) and run an algorithm for minimal red-blue dominating set.
  A solution may be repeated $O(n)$ times*.

[Kanté, Khoshkhah, Pourmoradnasseri, '18]
Minimal red-blue dominating set is tractable on (C_6, C_8)-free bipartite graphs.

Overview of the algorithm:

- If $|\,\text{red}(R)| \geq 2$, then we can list by brute-force in polynomial time!
- If $|\,\text{red}(R)| = |R \cap N(x)| = 1$, then we try all triplets (x, y, z) and run an algorithm for minimal red-blue dominating set.
  A solution may be repeated $O(n)$ times*.

What about the case where $|R \cap N(x)| \geq 2$?

A subset $Y \subseteq N(x)$, where $|Y| \geq 2$, is extendable if there exists a minimal redundant set R where $\text{red}(R) = \{x\}$ and $R \cap N(x) = Y$.

Lemma

The set Y is extendable with respect to x if and only if the two conditions hold:

(a) *Each $y \in Y$ has a private neighbor with respect to $Y \cup \{x\}$; and*

(b) *The set $N^2(x) \setminus N(Y)$ dominates $N(x) \setminus Y$.*

Lemma

If $Y^\star \subseteq N(x)$ is extendable with respect to x, then every subset $Y \subsetneq Y^\star$ satisfying $|Y| \geq 2$ is also extendable with respect to x.

It forms an independence set system!

Theorem (Castelo, Chalopin, Defrain, Vilmin, '26)

There exists:

- *a polynomial-delay algorithm that solves* GRAPHMRED·ENUM *on* (C_3, C_5, C_6, C_8)*-free graphs; and*
- *an incremental quasi-polynomial-time algorithm that solves* GRAPHMRED·ENUM *on* (C_3, C_5, C_6)*-free graphs.*

Some open questions:

- What is the complexity of GraphMirr·Enum and GraphMred·Enum in bipartite graphs?
- Does GraphMirr·Enum admit an output-polynomial-time algorithm for general graphs?

Thank you :)

References I

- Blind, Sarah, Nadia Creignou, Frédéric Olive (2021). "Locally definable vertex set properties are efficiently enumerable". In: *Discrete Applied Mathematics* 303, pp. 186–202.
- Bodlaender, Hans, Endre Boros, Pinar Heggernes, Dieter Kratsch (2015). "Open Problems of the Lorentz Workshop, "Enumeration Algorithms using Structure"". In.
- Boros, Endre, Kazuhisa Makino (2016). "Generating maximal irredundant and minimal redundant subfamilies of a given hypergraph". In: *WEPA 2016 & Boolean Seminar 2017*.
- — (2024). "Generating minimal redundant and maximal irredundant subhypergraphs". In: *Discrete Applied Mathematics* 358, pp. 217–229.
- Castelo, Emanuel, Jérémie Chalopin, Oscar Defrain, Simon Vilmin (2026). *Generating minimal redundant and maximal irredundant sets in incidence graphs*.

References II

- Conte, Alessio, Mamadou Moustapha Kanté, Andrea Marino, Takeaki Uno (2019). "Maximal Irredundant Set Enumeration in Bounded-Degeneracy and Bounded-Degree Hypergraphs". In: *International Workshop on Combinatorial Algorithms*. Springer, pp. 148–159.
- Kanté, Mamadou Moustapha, Kaveh Khoshkhah, Mozhgan Pourmoradnasseri (2018). "Enumerating minimal transversals of hypergraphs without small holes". In: *43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018)*. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, pp. 55–1.