Enumerating minimal dominating sets and variants in chordal bipartite graphs

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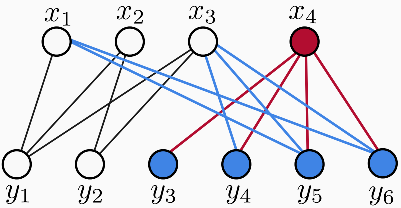
Definition (Weak-simplicial)

A vertex $v \in V(G)$ is **weak-simplicial** if

- $N(v)$ is an independent set; and
- for every pair $x, y \in N(v)$, either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$.

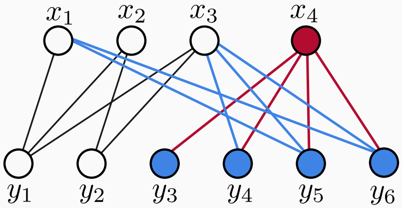


Definition (Weak elimination ordering)

An ordering $v_1, \ldots, v_n$ of $V(G)$ is said to be a **weak elimination ordering** if, for all $i \in [n]$, v_i is weak-simplicial in $G_i := G[\{v_1, \ldots, v_i\}]$.

Theorem ([1][2])

A graph is chordal bipartite if and only if it admits a weak elimination ordering.



[1] Uehara, "Linear time algorithms on chordal bipartite and strongly chordal graphs", 2002.
[2] Kurita et al., "An Efficient Algorithm for Enumerating Chordal Bipartite Induced Subgraphs in Sparse Graphs", 2019.

Definition (Dominating Set)

A set $D \subseteq V(G)$ is a dominating set of G if $N[D] = V(G)$.

Definition (Total Dominating Set)

A set $D \subseteq V(G)$ is a total dominating set of G if $N(D) = V(G)$.

Definition (Connected Dominating Set)

A set $D \subseteq V(G)$ is a connected dominating set of G if $N[D] = V(G)$ and $G[D]$ is connected.

In all cases, D is said to be minimal if it is inclusion-wise minimal.

Domination problems: enumeration edition

Minimal Dominating Set Enumeration (MinDom·Enum)
Input: A graph G.
Output: All minimal dominating sets of G.

Minimal Connected Dominating Set Enumeration (CMinDom·Enum)
Input: A graph G.
Output: All minimal connected dominating sets of G.

Minimal Total Dominating Set Enumeration (TMinDom·Enum)
Input: A graph G.
Output: All minimal total dominating sets of G.

Let $\mathcal{I}$ be an instance, $n = |\mathcal{I}|$ the size of the instance, and $d = |Sol(\mathcal{I})|$ be the number of solutions of $\mathcal{I}$.

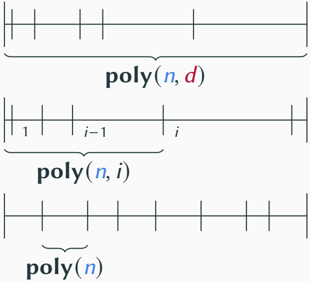
Total polynomial (**TotalP**)

$\supseteq$

Incremental polynomial (**IncP**)

$\supseteq$

Polynomial delay (**DelayP**)



What do we know?

- MinDom·Enum is in **IncP** on chordal bipartite graphs;[3]
- TMinDom·Enum is in **DelayP** on chordal bipartite graphs[3]; and
- CMinDom·Enum is MinDom·Enum-hard[4][5].

When restricted to chordal bipartite graphs, is

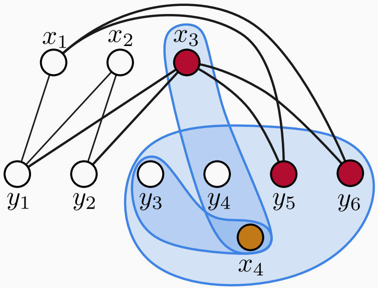
- **Question 1:** MinDom·Enum in **DelayP**?
- **Question 2:** CMinDom·Enum in **TotalP**?

[3] Golovach et al., "Enumerating minimal dominating sets in chordal bipartite graphs", 2016.
[4] Kanté et al., "On the enumeration of minimal dominating sets and related notions", 2014.
[5] The problem is conjectured to not admit a **TotalP** algorithm unless **P** = **NP**.

Observation

A **minimal dominating set** in a graph corresponds to a **minimal transversal** of the closed neighborhood hypergraph.

- Fix an ordering $(v_1, \ldots, v_n)$ of $V(G)$;
- Let $\mathcal{H}_i$ be the hypergraph of closed neighborhoods of G contained in $\{v_1, \ldots, v_i\}$;
- Extend $T \in Tr(\mathcal{H}_i)$ to $T' \in Tr(\mathcal{H}_{i+1})$.



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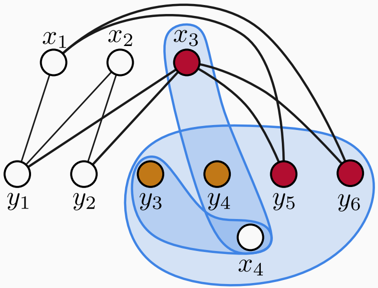

The algorithm: general idea

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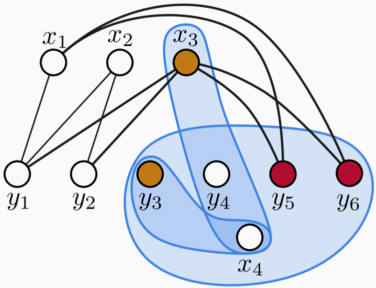
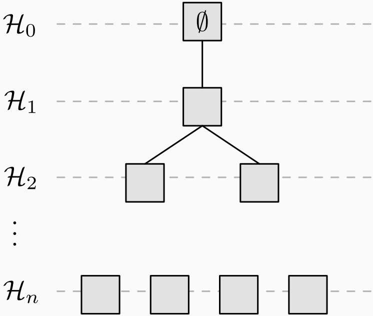


How to extend 101: the sequential method



What do we want? A search tree!

- No cycles; and
- No leaves of height $i < n$.

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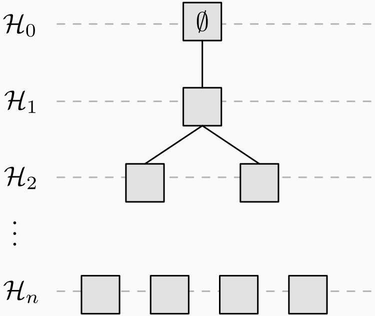

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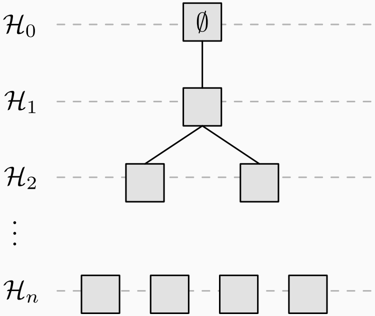
Parent of $T \in Tr(\mathcal{H}_{i+1})$ (⋆)

Repetedly remove the smallest vertex $v \in T$ with no private neighbors in $\mathcal{H}_i$.



What do we want? A search tree!

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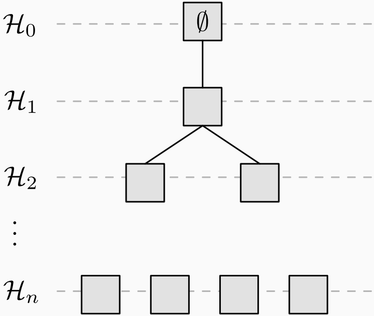
- No leaves of height $i < n$. ⋆

Ensuring children for $i \in [n-1]$ (⋆)

Let $T \in Tr(\mathcal{H}_i)$, where $i \in [n-1]$. Then, either

- $T \in Tr(\mathcal{H}_{i+1})$ and is its own parent; or
- $T \cup \{v_{i+1}\} \in Tr(\mathcal{H}_{i+1})$ and T is its parent.

How to extend 101: the sequential method



- $\Delta_{i+1} :=$ subset of $\mathcal{H}_{i+1}$ not hit by $T \in Tr(\mathcal{H}_i)$;

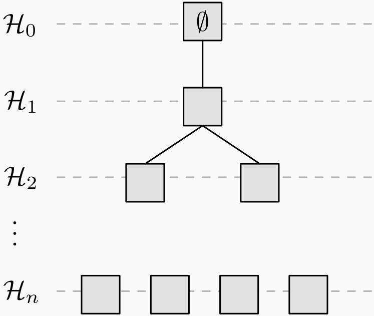
Lemma

If T is a child of $T^\star$, then $T = T^\star \cup X$, where $X \in Tr(\Delta_{i+1})$.

Observation

If $|Tr(\Delta_{i+1})|$ is polynomial and can be computed in polynomial time, then we have **DelayP + PSPACE**!

How to extend 101: the sequential method



- Δ_{i+1} := subset of $\mathcal{H}_{i+1}$ not hit by $T \in Tr(\mathcal{H}_i)$;
- **Bad news:** # of elements in $Tr(\Delta_{i+1})$ which are not children of T can be exponential;
- **Good news:** polynomial for chordal bipartite graphs!

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If T is a child of $T^\star$, then $T = T^\star \cup X$, where $X \in Tr(\Delta_{i+1})$.

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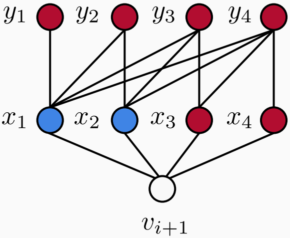
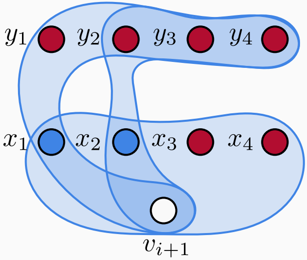
If $|Tr(\Delta_{i+1})|$ is polynomial and can be computed in polynomial time, then we have **DelayP + PSPACE**!

Observation

If v is weak-simplicial, then $G[N[v] \cup N^2(v)]$ is **bipartite chain**.

- $B := \{u \in N(v_{i+1}) \mid N[u] \in \Delta_{i+1}\}$;
- $R := \left(\bigcup_{H \in \Delta_{i+1}} H\right) \setminus B$.

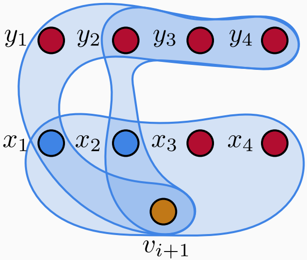




Lemma

Let $T \in Tr(\Delta_{i+1})$. Then

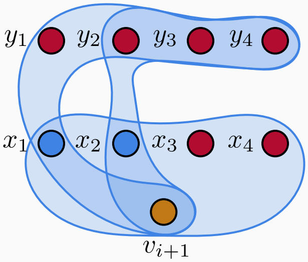
- $|T \cap R \cap N(v_{i+1})| \leq 1$*; and*
- $|T \cap N^2(v_{i+1})| \leq 1$.



Lemma

Let $T \in Tr(\Delta_{i+1})$. Then exactly one of the following holds:

- $T = \{v_{i+1}\}$;
- $T \subseteq B$, *in which case* $T = B$;
- $T \subseteq R$, *in which case* $|T| \leq 2$;
- $T = \{r\} \cup (B \setminus N(r))$ *for some* $r \in N^2(v_{i+1})$.



Results

We conclude that MinDom·Enum is in **DelayP + PSPACE** on chordal bipartite graphs.[6]

Some other results include:[6]

- CMinDom·Enum is in **IncP** on chordal bipartite graphs;
- The sequential method is **NP**-complete even if restricted bipartite graphs; and
- CMinDom·Enum is MinDom·Enum-hard even if restricted to bipartite graphs.[7]

[6] Castelo, Defrain, and Gomes, "Enumerating minimal dominating sets and variants in chordal bipartite graphs", 2025.
[7] Proved independently in: Kobayashi et al., "Enumerating minimal vertex covers and dominating sets with capacity and/or connectivity constraints", 2025.

Open questions

- Does MinDom·Enum admit a **DelayP** algorithm when restricted to bipartite graphs?
- Does CMinDom·Enum admit a **DelayP** algorithm when restricted to chordal bipartite graphs?
- Is CMinDom·Enum equivalent to MinDom·Enum?

Thank you!